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Quantum Ashkin–Teller Model Near the Decoupling Limit

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The critical properties and the spectrum of the quantum Ashkin–Teller chain are investigated by perturbation expansion around the exactly soluble Ising decoupling limit. The critical exponents are found to satisfy the hyperscaling relation and they are consistent with the conjectured values. The critical Hamiltonian in the finite-size scaling limit is transformed into a two-band spin-1/2 Fermi system, where the interaction energy in first order is the product of the band magnetizations. The complete spectrum is shown to exhibit a conformal structure with infinite primary operators, the anomalous dimensions of which are consistent with a Gaussian form.

KEY WORDS: Conformal invariance; Ashkin–Teller model; Gaussian operators; perturbation expansion; spin-1/2 Fermi system.

1. INTRODUCTION

The application of conformal invariance in the theory of critical phenomena has become extremely fruitful for two-dimensional classical and one-dimensional quantum systems.[1] The main developments of the theory have been achieved in two independent areas. The first important achievement is due to Belavin *et al.*[2] and Friedan *et al.*[3]: the complete classification of critical theories for conformal anomaly $c < 1$.

The other development of the theory is due to Cardy,[1,4] who obtained a powerful method to determine the anomalous dimensions of critical operators by using conformal mapping to relate the problem in the infinite plane to one in a simpler finite geometry. Let us define the critical Hamiltonian H of the two-dimensional classical system at the bulk critical

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point through the transfer matrix $T=\exp(-H)$. Then, according to conformal invariance,[4] the excitation spectrum of H on a periodic chain with N lattice sites has a towerlike structure in the finite-size scaling (FSS) limit[5]:

$$E_i - E_0 = \zeta(2\pi/N)(x+m+m') + O(1/N^2) \tag{1.1}$$

Here E_0 and E_i are the energy of the ground state and the ith excited state of H, respectively, x is the anomalous dimension of a primary operator, and m and m' are nonnegative integers. For two-dimensional classical models $\zeta = 1$, while for one-dimensional quantum models ζ is a normalizing factor, the so-called sound velocity.[6] There is a further relation between the FSS correction to the ground-state energy of the critical Hamiltonian with periodic boundary condition (BC) and the conformal anomaly number[7,8]:

$$E_0(N) = E_0(\infty) - (\pi/6N)\zeta c + O(1/N^2) \tag{1.2}$$

These relations (1.1)–(1.2) give a very efficient numerical and analytical tool to investigate the operator content (anomalous dimensions) of different critical models.

After classification of two-dimensional critical theories with $c<1$,[2,3] more attention has been paid to models with $c=1$.

Common features of these models are the coupling-dependent critical exponents and the presence of infinitely many primary operators among those also marginal ones. In the class of these models the exactly soluble Gaussian model[9] plays a central role, since many other models can be mapped into it using renormalization group arguments.[10] Now, a common belief is that in the $c=1$ critical theories all operators with coupling-dependent anomalous dimension are Gaussian-like.[9,10] This idea has been the starting point for conjecturing the operator content of different critical models (*XXZ* model,[11] AT model,[12–17] cubic model,[18] etc.).

To check the validity of these results, relations (1.1)–(1.2) can be successfully used. Analytical calculations can be performed for some models by the Bethe Ansatz method, since recently the method has been generalized to determine the FSS corrections to different energy levels.[19] In this way several complete conformal towers with Gaussian primary operators have been determined for the *XXZ* chain.[20] For some other models which may be mapped onto the *XXZ* chain with special boundary conditions some leading critical exponents have been determined. Among these models are the Ashkin–Teller (AT) model,[21–23] the Q-state Potts model,[21–23] and the $O(n)$ model.[24] Similar calculations have been performed for the eight-vertex model.[25]

Despite the new analytical results, there is now little hope of solving exactly the complete spectrum of a nontrivial interacting model along the critical line and in this way checking the above conjectures. Therefore it is of interest to determine the complete spectrum of a $c=1$ model even at one nonspecial point. In this paper we have done this exercise for the quantum AT model, performing a perturbation calculation around the Ising decoupling point. According to our results, the critical Hamiltonian can be transformed in the FSS limit into a two-band spin-1/2 fermion problem, where the interaction in first order is given by the product of the band magnetizations.

The setup of the paper is the following. In Section 2 the model is defined and the conjectured results are listed. In Section 3 the singularities of the thermodynamic quantities are determined and the hyperscaling relation is verified. In Section 4 the critical Hamiltonian of the model is expressed in terms of fermion operators and the leading critical exponents are determined. In Section 5 the critical Hamiltonian is diagonalized in first order of the interaction. Finally, Section 6 contains a summary, while the spectrum and the correlation functions of the reference system, the quantum Ising model, are presented in the Appendix. A short account of the results has already been published.[26]

2. THE MODEL

The quantum AT model,[27] similarly to the classical one, may be defined by two interacting quantum Ising models in the following form[28]:

$$H = H_0 + \lambda V \tag{2.1}$$

Here the unperturbed Hamiltonian is expressed in terms of two sets of Pauli matrices $\{s_i^x, s_i^z\}$, $\{\tau_i^x, \tau_i^z\}$:

$$H_0 = H_s + H_\tau = -\sum_{i=1}^{N} (s_i^x s_{i+1}^x + h s_i^z) - \sum_{i=1}^{N} (\tau_i^x \tau_{i+1}^x + h\tau_i^z) \tag{2.2}$$

while the perturbation contains four-spin and two-spin interactions between the s and τ spins:

$$V = -\sum_{i=1}^{N} s_i^x s_{i+1}^x \tau_i^x \tau_{i+1}^x - h \sum_{i=1}^{N} s_i^z s_i^z \tag{2.3}$$

The boundary conditions are chosen to be compatible with the torus; thus

$$s_{N+1}^x = g_s s_1^x \qquad \text{and} \qquad \tau_{N+1}^x = g_\tau \tau_1^x \tag{2.4}$$

with $|g_s| = |g_\tau| = 1$. For periodic BC, $g_s = g_\tau = 1$; for antiperiodic BC, $g_s = g_\tau = -1$; while for mixed BC, $g_s = -g_\tau$.

It is well known that the unperturbed Hamiltonian in (2.2) is diagonal in terms of fermion creation $(\eta^+_{k_s}, \mu^+_{k_\tau})$ and annihilation $(\eta_{k_s}, \mu_{k_\tau})$ operators.[29-31] According to the results recapitulated in the Appendix [Eq. (A5)],

$$H_s = \sum_{k_s} \Lambda_{k_s}(\eta^+_{k_s}\eta_{k_s} - 1/2), \qquad H_\tau = \sum_{k_\tau} \Lambda_{k_\tau}(\mu^+_{k_\tau}\mu_{k_\tau} - 1/2) \tag{2.5}$$

where the energy of the modes Λ_k is given by (A6). At the critical point, at $h^* = 1$, it is

$$\Lambda^*_k = 4\cos(k/2) \tag{2.6}$$

(In the following we use an asterisk to denote the value of the quantities at the critical point.)

The allowed sets of the k_s and k_τ numbers depend on the length of the chain, on the number of fermions N_s and N_τ in the s and τ subsystems, respectively, and on the form of the BC. They are from two sets; the possible values are listed in the Appendix [Eqs. (A9)–(A11)]. In the following we reexpress the interaction (2.3) in terms of fermion operators. Using the relations between spin and fermion variables (A17) and (A18), the interaction energy is easy to express as

$$V = -\sum D(k_1, k_2, k_3, k_4)(\eta^+_{k_1} - \eta_{k_1})(\eta^+_{k_2} + \eta_{k_2})(\mu^+_{k_3} - \mu_{k_3})(\mu^+_{k_4} + \mu_{k_4}) \tag{2.7}$$

where the summation runs over $k_1, k_2 \in \{k_s\}$ and $k_3, k_4 \in \{k_\tau\}$, and $D(k_1, k_2, k_3, k_4\}$ is defined in terms of the components of the eigenvectors Φ_k and Ψ_k [(A7), (A8)]:

$$D(k_1, k_2, k_3, k_4) = \sum_{i=1}^{N} (\Psi_{k_1,i}\Phi_{k_2,i+1}\Psi_{k_3,i}\Phi_{k_4,i+1} + \Psi_{k_1,i}\Phi_{k_2,i}\Psi_{k_3,i}\Phi_{k_4,i}) \tag{2.8}$$

At the critical point the $\Psi_{k,i}$ modes (A8) have the simple form

$$\Psi^*_{k,i} = -(1/N)^{1/2}\{\sin[k(i+1/2)] + \cos[k(i+1/2)]\} = -\Phi^*_{k,i+1/2} \tag{2.9}$$

With these functions the sum in (2.8) is easy to evaluate with the result

$$D^*(k_1, k_2, k_3, k_4)$$
$$= (1/N)\{\cos[(k_2+k_4)/2][f(k_1-k_2+k_3-k_4)$$
$$-f(k_1+k_2+k_3+k_4)]$$
$$+\cos[(k_2-k_4)/2][f(k_1-k_2-k_3+k_4)+f(k_1+k_2-k_3-k_4)]\} \tag{2.10}$$

Here $f(k)=\delta(k)-\delta(2\pi-k)-\delta(2\pi+k)$ is the sum of delta functions. We note for low excited states ($|k_s|$, $|k_\tau| \approx \pi$) $f(k)=\delta(k)$.

Closing this section, we briefly summarize the conjectured results for the model around the decoupling point. In the region $-1/\sqrt{2} \leqslant \lambda \leqslant 1$ a phase transition takes place in the system; the critical point is at $h^*=1$ independently of λ. Along the critical line the degeneracy of the ground state is increased by four. The critical operators are of two classes: operators with constant anomalous dimensions, and those with coupling-dependent critical exponents. In the later case the anomalous dimensions are assumed to have a Gaussian form[9,13]:

$$x = m + m' + (L+M\varepsilon)^2/(16\varepsilon) + (L-M\varepsilon)^2/(16\varepsilon) \tag{2.11}$$

with L and M nonnegative integers and ε is defined as

$$\varepsilon = \pi/[4\arccos(-\lambda)] \tag{2.12}$$

The anomalous dimensions of the leading operators (energy, magnetization, and polarization) are[12,27]

$$\begin{aligned} x_\varepsilon &= \pi/[2\arccos(-\lambda)] \\ x_m &= 1/8 \\ x_p &= x_\varepsilon/4 \end{aligned} \tag{2.13}$$

The sound velocity of the model is conjectured as[12]

$$\zeta = \pi\sin(\arccos\lambda)/\arccos\lambda \tag{2.14}$$

and the conformal anomaly is $c=1$.

3. PERTURBATION EXPANSION FOR THE THERMODYNAMIC QUANTITIES

Different perturbation methods are widely used to determine thermodynamic properties of complicated systems. These methods, however, are usually not applicable at the critical point due to the lack of information on the correlation properties of the reference system at criticality. An appropriate model for such a purpose is the two-dimensional Ising model (and its one-dimensional quantum version). An expansion around an Ising limit was first performed by Kadanoff and Wegner[32] for the eight-vertex model. Later the method was applied for the classical AT model[33,34] and for generalized Ising models.[35] In this paper we perform the perturbation calculation for the quantum AT model around the Ising decoupling limit. In contrast to the previous work,[32-35] we determine not only the singularities of the thermodynamic quantities (Section 3), but the spectrum of the critical Hamiltonian in the FSS limit as well (Sections 4 and 5).

3.1. Ground-State Energy

The ground-state energy per site near the decoupling limit may be written as

$$E_0/N = -2\langle 0|s_1^x s_2^x|0\rangle - 2h\langle 0|s_1^z|0\rangle - \lambda[\langle 0|s_1^x s_2^x|0\rangle^2 + h\langle 0|s_1^z|0\rangle^2] + O(\lambda^2) \tag{3.1}$$

where $\langle 0|\cdots|0\rangle$ denotes the average in the ground state of the quantum Ising model. E_0/N can be expressed in closed form using (A19):

$$E_0/N = -(4/\pi)(1+h)\,E(h') - \lambda 4/\pi^2\{E^2(h) + (1/h)[E(h) + (h^2-1)\,K(h)]^2\} + O(\lambda^2) \tag{3.2}$$

where $K(x)$ and $E(x)$ are the complete elliptic integrals of the first and second kinds [(A14), (A20)] and $h' = 2(h)^{1/2}/(1+h)$. The E_0 is nonanalytic for $h = 1$. Using the asymptotic expansions[36]

$$\begin{aligned} \lim_{x\to 1} E(x) &= 1 - (1/2)(1-x)[\log|1-x| - 2\log(8)] + \cdots \\ \lim_{x\to 1} K(x) &= \log(2\sqrt{2}) - (1/2)\log|1-x| + \cdots \end{aligned} \tag{3.3}$$

one can expand the ground-state energy

$$E_0/N = -(8/\pi)(1+\lambda/\pi) + (1/\pi)(1-h)^2\log|1-h|\,[1 - \lambda(2/\pi)\log|1-h|] + \cdots \tag{3.4}$$

This is equivalent to the singular behavior

$$\lim_{h \to 1} E_0^{\text{sing}}/N \sim |h-1|^{2-\alpha(\lambda)} \tag{3.5}$$

with the specific heat exponent

$$\alpha(\lambda) = (4/\pi)\lambda + O(\lambda^2) \tag{3.6}$$

3.2. The Energy Gap

The first excited state of the AT model is twofold degenerate at $\lambda = 0$; it may be obtained if the "τ" system is in the ground state and the "s" system is in the first excited one and vice versa. This degeneracy is preserved along the critical line. The gap around $\lambda = 0$ may be written as

$$\begin{aligned} E_1 - E_0 = & -N\{[\langle 1| s_1^x s_2^x |1\rangle - \langle 0| s_1^x s_2^x |0\rangle \\ & + h(\langle 1| s_1^z |1\rangle - \langle 0| s_1^z |0\rangle)] \\ & + \lambda[\langle 1| s_1^x s_2^x |1\rangle \langle 0| s_1^x s_2^x |0\rangle - \langle 0| s_1^x s_2^x |0\rangle^2 \\ & + h[\langle 1| s_1^z |1\rangle \langle 0| s_1^z |0\rangle - \langle 0| s_1^z |0\rangle^2]\} + O(\lambda^2) \end{aligned} \tag{3.7}$$

Using (A19) and (A21), one can express the gap for $h > 1$ as

$$E_1 - E_0 = 2(h-1)[1 + \lambda(2/\pi)(h+1)\, K(h)] + O(\lambda^2) \tag{3.8}$$

which according to (3.3) around $h = 1$ behaves as

$$\lim_{h \to 1} (E_1 - E_0) = 2(h-1)[1 - \lambda(2/\pi) \log|h-1|] + \cdots \tag{3.9}$$

Thus, the gap exponent $\nu(\lambda)$ (equal to the correlation length exponent) defined as

$$\lim_{h \to 1} (E_1 - E_0) \sim 2(h-1)^{\nu(\lambda)} \tag{3.10}$$

is given in first order by

$$\nu(\lambda) = 1 - \lambda(2/\pi) + O(\lambda^2) \tag{3.11}$$

Comparing the critical exponents in (3.6) and (3.11), we can conclude that they satisfy the hyperscaling relation $2 - \alpha(\lambda) = d\nu(\lambda)$ with $d = 2$. Furthermore, the $x_\varepsilon = d - 1/\nu$ anomalous dimension agrees to linear order with the conjectured value in (2.7).

4. PERTURBATION EXPANSION FOR THE CRITICAL HAMILTONIAN

4.1. Introduction

In this section the perturbation calculation is performed along the critical line of the model, but for finite chains. In this way the anomalous dimensions of the leading operators will be determined through Eq. (1.1) and the validity of conformal invariance for our model will be checked at the same time.

The critical Hamiltonian has been expressed in terms of fermion operators by Eqs. (2.5), (2.7), and (2.10). Obviously the effect of V is different if $\{k_s\}$ and $\{k_\tau\}$ are the same set or they are different. In the latter case the $\eta_{k_s}^+|0\rangle$ and $\mu_{k_\tau}^+|0\rangle$ states have different energies; consequently, no mixing takes place between the states of the two subsystems. In these sectors at $\lambda=0$ the Hamiltonian has no special symmetry; thus, the degeneracy of the levels and the values of the anomalous dimensions are coupling independent: they can be obtained from the Ising values at $\lambda=0$. (The λ dependence of the gaps in the FSS limit is purely due to the sound velocity.)

The situation is more complicated if $\{k_s\}$ and $\{k_\tau\}$ are the same set; thus, the s and τ subsystems are indistinguishable. The consequence is mixing of states of the two subsystems, occurrence of an exchange energy, and splitting of some $\lambda=0$ degenerate levels. In the following these sectors will be investigated.

Since the two subsystems are indistinguishable, many levels at $\lambda=0$ are degenerate; thus, a degenerate perturbation calculation have to be performed. In the following we collect those terms of V (denoted by V_{d}) which act between degenerate states. Obviously V_{d} contains the fermion operators in three different possible combinations: (i) $\eta_k^+\eta_k\mu_{k'}^+\mu_{k'}$, (ii) $\eta_k^+\eta_{k'}^+\mu_k\mu_{k'}$, and (iii) $\eta_k^+\eta_{k'}\mu_{k'}^+\mu_k$. In the following we determine the interactions belonging to the different processes.

(i) The diagonal term is characterized by the number $k_1=k_2=k$ and $k_3=k_4=k'$. One obtains for this interaction

$$
\begin{aligned}
V_1 = &-(8/N)\left[\sum_k \cos(k/2)(\eta_k^+\eta_k-1/2)\right] \\
&\times\left[\sum_{k'} \cos(k'/2)(\mu_{k'}^+\mu_{k'}-1/2)\right] \\
&-(4/N)\sum_{k,k'}(\delta_{k,k'}-\delta_{k,-k'})(\eta_k^+\eta_k-1/2)(\mu_{k'}^+\mu_{k'}-1/2) \qquad (4.1)
\end{aligned}
$$

Here the first term is proportional to the product of the Hamiltonians of the subsystems (2.5):

$$V_{\text{cn}} = -(1/2N)\, H_s H_\tau \tag{4.2}$$

while the second term is due to the interference effect:

$$V^1_{\text{sp}} = -(4/N) \sum_k (\eta_k^+ \eta_k - 1/2)(\mu_k^+ \mu_k - 1/2) - (\eta_k^+ \eta_k - 1/2)(\mu^+_{-k} \mu_{-k} - 1/2) \tag{4.3}$$

(ii) In the second process two fermions are created in one subsystem and annihilated at the same time in the other. This interaction is given as

$$V^2_{\text{sp}} = (4/N) \sum_{k>k'} \{[1 - \cos(k/2)\cos(k'/2)]\, \eta_k^+ \eta_{k'}^+ \mu_k \mu_{k'} + \sin(k/2)\sin(k'/2) \eta_k^+ \eta_{k'}^+ \mu_{-k} \mu_{-k'}\} + \{\eta \to \mu,\ \mu \to \eta\} \tag{4.4}$$

where the restriction $k > k'$ is to avoid double counting, and the notation $\{\eta \to \mu,\ \mu \to \eta\}$ means that in the first expression the characters η and μ simultaneously have to be interchanged.

(iii) In the third process fermions with k and k' numbers are exchanged in the two subsystems. This interaction is given in the following form:

$$V^3_{\text{sp}} = (4/N) \sum_{k>k'} \{[\sin(k/2)\sin(k'/2) - \delta_{k,-k'} \cos^2(k/2)]\, \eta_k^+ \eta_{k'} \mu^+_{-k'} \mu_{-k} - [1 + \cos(k/2)\cos(k'/2)(1 - \delta_{k,-k'})]\, \eta_k^+ \eta_{k'} \mu_{k'}^+ \mu_k\} + \{\eta \to \mu,\ \mu \to \eta\} \tag{4.5}$$

Collecting the different contributions gives the effective interaction as

$$V_{\text{d}} = V_{\text{cn}} + V_{\text{sp}} \tag{4.6}$$

with

$$V_{\text{sp}} = V^1_{\text{sp}} + V^2_{\text{sp}} + V^3_{\text{sp}} \tag{4.7}$$

Here V_{cn} is also present if $\{k_s\}$ and $\{k_\tau\}$ are different, and as will be shown later, in the FSS limit it produces the same shift on degenerate levels and determines the sound velocity of the model. On the other hand, V_{sp} is due to the interference effect and is present only if $\{k_s\}$ and $\{k_\tau\}$ are the same.

This term splits the degenerate levels and produces coupling-dependent critical exponents.

In the following we calculate the low-lying energy levels to linear order in λ for periodic BC.

4.2. λ Correction to the Low-Lying Levels for Periodic BC

4.2.1. Ground-State Energy. The ground state of H_0, denoted by $|0\rangle$, is the η and μ particle vacuum. The only correction to this state is due to $V_{\rm cn}$ [Eq. (4.2)] and the ground-state energy along the critical line is given by

$$E_0^* = -E_0^{P*} - (\lambda/2N)[E_0^{P*}]^2 + O(\lambda^2) \tag{4.8}$$

which may be written, using (A15), as

$$\begin{aligned} E_0^* &= -4\,\mathrm{cosec}(\pi/2N) - \lambda(2/N)\,\mathrm{cosec}^2(\pi/2N) + O(\lambda^2) \\ &= -N(8/\pi)(1+\lambda/\pi) - (\pi/6N)\,\zeta(\lambda) + \cdots \end{aligned} \tag{4.9}$$

Here

$$\zeta(\lambda) = 2 + \lambda(4/\pi) + O(\lambda^2) \tag{4.10}$$

denotes a normalizing factor.

In the following we show that $\zeta(\lambda)$ measures the distance between the equidistant levels of the conformal tower (1.1); thus, it is the sound velocity of the model. Let us consider a nondegenerate level (for example, in the $\{k_s\} \neq \{k_\tau\}$ sectors) with excitation energy $\Delta_0 = \Delta_s + \Delta_\tau$. Then the gap of the perturbed system will be changed due to $V_{\rm cn}$ [Eq. (4.2)] as follows:

$$\begin{aligned} \Delta &= \Delta_s + \Delta_\tau - (\lambda/2N)[(E_0^{P*} + \Delta_s)(E_0^{P*} + \Delta_\tau) - (E_0^{P*})^2] + O(\lambda^2) \\ &= (\Delta_s + \Delta_\tau)[1 - (\lambda/2N)E_0^{P*}] + O(\lambda^2, N^{-2}) \end{aligned} \tag{4.11}$$

Here we use the fact that for low-lying excitations according to Eq. (1.1) $\Delta_s = (2\pi/N)2(x_s + m_s)$ and $\Delta_\tau = (2\pi/N)2(x_\tau + m_\tau)$; thus,

$$\Delta = (2\pi/N)(x_s + x_\tau + m_s + m_\tau)[2 + (4/\pi)\lambda] + \cdots \tag{4.12}$$

Consequently, the sound velocity is given by (4.10), which agrees to first order with the conjectured value in (2.14).

Furthermore, comparing (4.9) with (1.2), we can conclude that the conformal anomaly of the model is $c = 1 + O(\lambda^2)$, as expected. In the following the first gaps of the different sectors will be determined.

4.2.2. Magnetization Sector (N_s odd, N_τ even or vice versa). The first excited state is twofold degenerate for $\lambda=0$; they are $\mu_\pi^+|0\rangle$ and $\eta_\pi^+|0\rangle$, but there is no mixing between these two states. Thus, the states remain degenerate for $\lambda\neq 0$, too, with the gap

$$\Delta_{1,2}^m = 2\,\mathrm{tg}(\pi/4N) + (\lambda/N)\cos^{-2}(\pi/4N) = (2\pi/N)\,\zeta(\lambda)(1/8) + \cdots \tag{4.13}$$

Comparing this result with (1.1), we can conclude that the anomalous dimension of the magnetization operator is $x_m = 1/8$, in accordance with the conjectured result (2.13).

4.2.3. Polarization Sector (N_s odd, N_τ odd). The first excited state is nondegenerate for $\lambda=0$; it is given by $\eta_\pi^+\mu_\pi^+|0\rangle$. The corresponding gap

$$\Delta_1^P = 4\,\mathrm{tg}(\pi/4N) + O(\lambda^2) \tag{4.14}$$

is independent of λ in first order. Normalizing with the sound velocity (4.10), one obtains

$$\Delta_1^P = (2\pi/N)(1/4)[1-\lambda(2/\pi)]\,\zeta(\lambda) + \cdots \tag{4.15}$$

Thus, the anomalous dimension of the polarization operator is

$$x_P = (1/4)[1-\lambda(2/\pi)] + O(\lambda^2) \tag{4.16}$$

in accordance with (2.13).

4.2.4. Energy Sector (N_s odd, N_τ odd). The first excited state is twofold degenerate; for $\lambda=0$ they are $\eta_{k_1}^+\eta_{-k_1}^+|0\rangle$ and $\mu_{k_1}^+\mu_{-k_1}^+|0\rangle$ with $k_1=\pi-\pi/N$. These states are mixed by V_{sp} and the secular determinant determining the energy perturbation $\lambda E_{1,2}$ is given by

$$\begin{vmatrix} A-E_{1,2} & B \\ B & A-E_{1,2} \end{vmatrix} = 0 \tag{4.17}$$

with

$$\begin{aligned} A &= -(E_0^{P*}/2N)[E_0^{P*} + 4\cos(k_1/2)] \\ B &= (8/N)\sin^2(k_1/2) \end{aligned} \tag{4.18}$$

and with the solution $E_{1,2} = A \pm B$. Then the first two gaps in the energy sector are given by

$$\begin{aligned} \Delta_{1,2}^e &= 8\sin(\pi/2N) + \lambda(8/N)[1\pm\cos^2(\pi/2N)] \\ &= (2\pi/N)\,\zeta(\lambda)[1\pm\lambda(2/\pi)] + \cdots \end{aligned} \tag{4.19}$$

Thus, the anomalous dimension of the energy operator is

$$x_e = 1 - \lambda(2/\pi) + O(\lambda^2) \tag{4.20}$$

in accordance with the conjectured result (2.13).

To illustrate the appearence of conformal towers, we calculate the next levels in the energy sector. The second excited state is eightfold degenerate for $\lambda = 0$, and these states belong to two orthogonal subspaces for $\lambda \neq 0$ spanned by the vectors

$$\eta_{k_1}^+ \eta_{k_2}^+ |0\rangle, \quad \eta_{-k_1}^+ \eta_{-k_2}^+ |0\rangle, \quad \mu_{k_1}^+ \mu_{k_2}^+ |0\rangle, \quad \mu_{-k_1}^+ \mu_{-k_2}^+ |0\rangle$$

and

$$\eta_{k_1}^+ \eta_{-k_2}^+ |0\rangle, \quad \eta_{k_2}^+ \eta_{-k_1}^+ |0\rangle, \quad \mu_{k_1}^+ \mu_{-k_2}^+ |0\rangle, \quad \mu_{k_2}^+ \mu_{-k_1}^+ |0\rangle$$

with $k_2 = \pi - 3\pi/N$. For $\lambda \neq 0$ both sets of four levels split, with the following gaps:

$$\begin{aligned} \Delta_{3,4}^e &= 4d_1 + 4d_2 + \lambda(4/N)\{1 + d_2/d_1 \pm [1 + \cos(2\pi/N)]\} \\ \Delta_{5,6}^e &= 4d_1 + 4d_2 + \lambda(4/N)\{1 + d_2/d_1 \pm [1 - \cos(\pi/N)]\} \end{aligned} \tag{4.21}$$

where $d_i = \cos(k_i/2)$. In the FSS limit these gaps behave as

$$\Delta_{3,4,5,6}^e = (2\pi/N)\, \zeta(\lambda) \begin{cases} 2 \pm \lambda 0 \\ 2 \pm \lambda(2/\pi) \end{cases} \tag{4.22}$$

and all these levels are two-fold degenerate. We can see that the exponents given in the first row of (4.22) belong to new primary operators, while those in the second row are elements of the conformal towers of the primary operators in (4.19). To determine the gaps for other excited states, one can proceed in a similar way. The calculation, however, become extremly complicated for such levels, which are highly degenerate, since high-dimensional secular matrices have to be diagonalized. We show in the next section that this task can be done analytically in the FSS limit.

5. DIAGONALIZATION OF THE HAMILTONIAN IN THE FSS LIMIT

In the FSS limit only $O(1/N)$ terms of the Hamiltonian are important and we restrict ourselves to low-lying fermion states, i.e., for which $|k| = \pi - m\pi/N$, $m \ll N$, and $|\sin(k/2)| = 1 + O(1/N^2)$, $\cos(k/2) = m\pi/2N$. In this limit the coefficients of V_{sp} in (4.4)–(4.5) do not depend on k and k' and these equations are reduced to

$$\tilde{V}^2_{sp} = (4/N) \sum_{k<k'} [\eta_k^+ \eta_{k'}^+ \mu_k \mu_{k'} + \text{sign}(kk') \eta_k^+ \eta_{k'}^+ \mu_{-k} \mu_{-k'}] + \{\eta \to \mu, \mu \to \eta\} \quad (5.1)$$

$$\tilde{V}^3_{sp} = (4/N) \sum_{k<k'} [-\eta_k^+ \eta_{k'} \mu_{k'}^+ \mu_k + \text{sign}(kk') \eta_k^+ \eta_{k'} \mu_{-k'}^+ \mu_{-k}] + \{\eta \to \mu, \mu \to \eta\} \quad (5.2)$$

The sum of these terms together with V^1_{sp} in (4.3) can be written in the following form:

$$\tilde{V}_{sp} = (4/N) \sum_{k,k'} V_{k,k'} \quad (5.3)$$

where

$$\begin{aligned} V_{k,k'} = (1/2)[&\eta_k^+ \eta_{k'}^+ \mu_k \mu_{k'} \\ &+ \text{sign}(kk') \eta_k^+ \eta_{k'}^+ \mu_{-k} \mu_{-k'} + \mu_k^+ \mu_{k'}^+ \eta_k \eta_{k'} \\ &+ \text{sign}(kk') \mu_k^+ \mu_{k'}^+ \eta_{-k} \eta_{-k'}] - \eta_k^+ \eta_{k'} \mu_{k'}^+ \mu_k \\ &+ \text{sign}(kk') \eta_k^+ \eta_{k'} \mu_{-k'}^+ \mu_{-k} \end{aligned} \quad (5.4)$$

and there is no restriction on k and k'.

Now let us collect terms with the same absolute values of k and k' and define

$$\hat{V}_{k,k'} = V_{k,k'} + V_{-k,k'} + V_{k,-k'} + V_{-k,-k'} \quad (5.5)$$

for $\pi > k$, $k' > 0$. (The $k = \pi$ and $k = 0$ terms will be considered separately.) In terms of the fermion operators

$$a_k^+ = (1/\sqrt{2})(\eta_k^+ + i\eta_{-k}^+), \qquad b_k^+ = (1/\sqrt{2})(\mu_k^+ + i\mu_{-k}^+) \quad (5.6)$$

$\hat{V}_{k,k'}$ has a simple form:

$$\begin{aligned} \hat{V}_{k,k'} &= -[a_k^+ b_k + a_k b_k^+][a_{-k'}^+ b_{-k'} + a_{-k'} b_{-k'}^+] \\ &\quad - [a_{k'}^+ b_{k'} + a_{k'} b_{k'}^+][a_{-k}^+ b_{-k} + a_{-k} b_{-k}^+] \\ &= e_k e_{-k'} + e_{k'} e_{-k} \end{aligned} \quad (5.7)$$

Here e_k is defined as

$$e_k = -i(a_k^+ b_k + a_k b_k^+) \tag{5.8}$$

and it is diagonal

$$e_k = g_k^+ g_k - h_k^+ h_k \tag{5.9}$$

in terms of the new Fermi operators:

$$g_k^+ = (1/\sqrt{2})(a_k^+ + ib_k^+), \qquad h_k^+ = (1/\sqrt{2})(a_k^+ - ib_k^+) \tag{5.10}$$

Thus, the effektive interaction (5.3) is also diagonal:

$$\tilde{V}_{\rm sp} = (4/N) \sum_{\pi > k, k' > 0} (e_k e_{-k'} + e_{k'} e_{-k}) = (8/N) \left(\sum_{k>0} e_k \right) \left(\sum_{k>0} e_{-k} \right) \tag{5.11}$$

The $\sum_k e_k$ operator may be considered as a magnetization operator if we assign spin up for the $g_k |0\rangle$ fermions and spin down for the $h_k |0\rangle$ fermions. Let us introduce spin-1/2 fermion operators with the following definitions:

$$\begin{aligned} c_{k,\uparrow}^+ &= g_k^+, \qquad c_{k,\downarrow}^+ = h_k^+ \\ d_{k,\uparrow}^+ &= g_{-k}^+, \qquad d_{k,\downarrow}^+ = h_{-k}^+, \qquad \pi \geqslant k \geqslant 0 \end{aligned} \tag{5.12}$$

In this way two new subsystems have been introduced, whose Hamiltonians are given by

$$H_c = \sum_{k,\sigma} \tilde{\Lambda}_k (c_{k,\sigma}^+ c_{k,\sigma} - 1/2), \qquad H_d = \sum_{k,\sigma} \tilde{\Lambda}_k (d_{k,\sigma}^+ d_{k,\sigma} - 1/2) \tag{5.13}$$

where the summation runs for $\pi \geqslant k \geqslant 0$ and $\sigma = \pm 1$. The energy of modes is $\tilde{\Lambda}_k = \Lambda_k$ for $k \neq 0$, and $\tilde{\Lambda}_0 = 2$ to avoid double counting. Obviously $H_0 = H_c + H_d$. Then we define magnetization operators for the subsystems as

$$M_c = \sum_{k,\sigma} \sigma_k c_{k,\sigma}^+ c_{k,\sigma}, \qquad M_d = \sum_{k,\sigma} \sigma_k d_{k,\sigma}^+ d_{k,\sigma} \tag{5.14}$$

where $\sigma_k = \pm 1$ for $\pi > k > 0$, but $\sigma_0 = \sigma_\pi = \pm 1/2$.

The effective Hamiltonian is diagonal in terms of these variables in the FSS limit:

$$\begin{aligned} H_0 &= H_c + H_d \\ V_{\rm cn} &= -(1/2N)\, H_c H_d + O(1/N^2) \\ V_{\rm sp} &= (8/N)\, M_c M_d + O(1/N^2) \end{aligned} \tag{5.15}$$

This is the main result of our paper. It means that the Ashkin–Teller model near the decoupling limit may be transformed into a two-band spin-1/2 fermion system, where the interaction energy is the product of the band magnetizations.

Now let us distribute the states into the different sectors, i.e., let us construct eigenstates of the fermion number operators $\exp(i\pi N_s)$ and $\exp(i\pi N_\tau)$. We note that the operator

$$T = T_c T_d = \prod_{k_c} \sigma^x_{k_c} \prod_{k_d} \sigma^x_{k_d} \tag{5.16}$$

commutes with the Hamiltonian; thus, the eigenstates of H may be classified as $T\varphi^{(+)} = \varphi^{(+)}$ and $T\varphi^{(-)} = -\varphi^{(-)}$. If φ is not an eigenstate of T, then $(T\varphi \pm \varphi)$ is an eigenstate with eigenvalues ± 1. Now using the definitions (5.6), (5.10), and (5.12), it is easy to show, that $\varphi^{(\pm)}$ is an $\exp(i\pi N_\tau) = \pm 1$ eigenstate.

Closing this section, we comment on our main result in (5.15). The perturbation is in diagonal form and consists of two parts. $V_{\rm cn}$ produces the same shift on degenerate levels and, as already mentioned, determines the sound velocity (4.10) of the model. The other part of the perturbation, $V_{\rm sp}$, splits the degenerate levels for $\lambda \neq 0$. This splitting energy depends only on the distribution of the fermions in the two possible spin states, but does not depend on the energy of the level at $\lambda = 0$. The consequence is the conformal tower structure of the spectrum (1.1) even for $\lambda \neq 0$. Due to the presence of $V_{\rm sp}$, nontrivial primary operators appear for $\lambda \neq 0$. According to (5.15), the number of primary operators is infinite and their anomalous dimensions are consistent with the Gaussian values (2.11). Up to linear order of λ they may be expressed as

$$x_i = n + n' + [(M/2)^2 + (L/4)^2] + 2\lambda/\pi[(M/2)^2 - (L/4)^2] + O(\lambda^2) \tag{5.17}$$

with n, n', M, L nonnegative integers.

6. SUMMARY

In this paper the critical properties and the spectrum of the quantum Ashkin–Teller model have been studied around the Ising decoupling limit. The Hamiltonian of the model is reexpressed in terms of fermion creation and annihilation operators and a linear-order perturbation calculation is performed using the spectrum and correlation functions of the quantum Ising model. The ground-state energy is found to be singular at $h^* = 1$ and the specific heat exponent (3.6) to be coupling dependent. The correlation length exponent (3.11) is also coupling dependent and these exponents obey the hyperscaling relation.

The critical Hamiltonian is studied for finite systems with boundary conditions compatible with the torus. The critical Hamiltonian is exactly diagonalized in the FSS limit by transforming it into a two-band spin-1/2 fermion system, where the interaction in linear order of λ is the product of the band magnetizations. The model is shown to be conformally invariant, i.e., the amplitudes of the scaling functions relate to the anomalous dimensions of given critical operators satisfying Eq. (1.1). The complete spectrum of the model consists of infinite conformal towers, where the anomalous dimensions of the critical operators obey the Gaussian form (5.17).

Thus, we can conclude that in this paper we have presented the first example for the complete spectrum of an interacting $c=1$ nontrivial model, whose spectrum in the finite-size scaling limit is in accordance with the statements of conformal invariance and with the conjectures of the mapping into the Gaussian model.

APPENDIX. EXCITATION SPECTRUM AND CORRELATIONS OF THE QUANTUM ISING MODEL

In this Appendix the known exact results on the quantum Ising model are recapitulated.

The quantum Ising model defined by the Hamiltonian

$$H_s = -\sum_{i=1}^{N} (s_n^x s_{n+1}^x + h s_n^z) \tag{A1}$$

may be transformed in terms of fermion creation and annihilation operators C_n^+ and C_n into the quadratic form[29,30]

$$\begin{aligned} H_s = &-\sum_{n=1}^{N-1} (C_n^+ - C_n)(C_{n+1}^+ + C_{n+1}) - 2h \sum_{n=1}^{N} (C_n^+ C_n - 1/2) \\ &+ g_s \exp(i\pi N_s)(C_N^+ - C_N)(C_1^+ + C_1) \end{aligned} \tag{A2}$$

where g_s is defined in (2.4) and

$$N_s = \sum_{n=1}^{N} C_n^+ C_n \tag{A3}$$

is the number of fermions. Through the canonical transformation

$$\eta_k = \sum_{n=1} \{C_n(\Phi_{kn} + \Psi_{kn})/2 + C_n^+(\Phi_{kn} - \Psi_{kn})/2\} \tag{A4}$$

H_s may be reexpressed in the diagonal form

$$H_s = \sum_k \Lambda_k(\eta_k^+ \eta_k - 1/2) \tag{A5}$$

with

$$\begin{aligned} \Lambda_k &= 2(1 + h^2 + 2h \cos k)^{1/2}, \qquad & k \neq \pi \\ \Lambda_k &= 2(h - 1), & k = \pi \end{aligned} \tag{A6}$$

The normal modes are given by

$$\Phi_{kn} = (1/N)^{1/2}[\sin(kn) + \cos(kn)] \tag{A7}$$

and

$$\begin{aligned} \Psi_{kn} &= -(2/\Lambda_k)(h\Phi_{kn} + \Phi_{k,n+1}), \qquad n \neq N \\ \Psi_{kN} &= -(2/\Lambda_k)[h\Phi_{kN} - g_s \exp(i\pi N_s)\Phi_{k,1}] \end{aligned} \tag{A8}$$

We note that (for $k \neq 0$ and $k \neq \pi$) the $\pm k$ modes are degenerate; thus, these eigenvectors are not uniquely determined. The advantage of the form used in (A7) is that it is a continuous function of k. The usual choice[30] for Φ_{kn} is a piecewise function of k and it is inconvenient for a perturbation calculation. The price one has to pay for the simpler form (A7) is the mixing of the momentum eigenstates.

The allowed set of the k numbers depends on N_s, on the form of the BC, and on the length of the chain. In the following, for simplicity we restrict ourselves to odd N values. Then the possible modes are form two sets[31]:

$$\omega = \{\pm\pi/N, \pm 3\pi/N, \ldots, \pm(N-1)\pi/N\} \tag{A9}$$

and

$$\vartheta = \{0, \pm 2\pi/N, \pm 4\pi/N, \ldots, \pm(N-2)\pi/N, \pi\} \tag{A10}$$

The allowed wavenumbers are the following:

(a) N_s even

$$k_{\mathrm{P}} \in \omega \qquad \text{and} \qquad k_{\mathrm{AP}} \in \vartheta \tag{A11a}$$

(The subscripts P and AP refer to periodic BC and to antiperiodic BC, respectively.)

(b) N_s or N_τ odd:

$$k_{\mathrm{P}} \in \vartheta \qquad \text{and} \qquad k_{\mathrm{AP}} \in \omega \tag{A11b}$$

The ground state of H_s is the fermion vacuum; thus, the ground-state energy is given by

$$E_0 = -(1/2) \sum_k \Lambda_k \tag{A12}$$

which may be expressed in the thermodynamic limit as

$$E_0/N = -(2/\pi)(1+h)\, E(2\sqrt{h}/(1+h)) \tag{A13}$$

where

$$E(x) = \int_0^{\pi/2} (1 - x^2 \sin^2 \alpha)^{1/2}\, d\alpha \tag{A14}$$

is the complete elliptic integral of the second kind. The ground-state energy for finite systems at $h^* = 1$ at the critical point is given by

$$E_0^{\mathrm{P}*} = -2 \operatorname{cosec}(\pi/2N) \tag{A15}$$

for periodic BC, and by

$$E_0^{\mathrm{AP}*} = -2 \cot(\pi/2N) \tag{A16}$$

for antiperiodic BC. The excited states of H_s may be constructed by creating fermions with wavenumbers given by (A11). To determine the nearest neighbor correlations, one have to express the Pauli matrices with fermion operators:

$$\begin{aligned} s_n^x s_{n+1}^x &= (C_n^+ - C_n)(C_{n+1}^+ + C_{n+1}) \\ &= \left[\sum_k \Psi_{kn}(\eta_k^+ - \eta_k)\right]\left[\sum_{k'} \Phi_{k',n+1}(\eta_{k'}^+ + \eta_{k'})\right] \end{aligned} \tag{A17}$$

and similarly

$$s_n^z = (C_n^+ - C_n)(C_n^+ + C_n) = \left[\sum_k \Psi_{kn}(\eta_k^+ - \eta_k)\right]\left[\sum_{k'} \Phi_{k',n}(\eta_{k'}^+ + \eta_{k'})\right] \tag{A18}$$

The averages of these expressions in the ground state may be expressed as

$$\begin{aligned} \langle 0 | s_n^x s_{n+1}^x | 0 \rangle &= (2/\pi)\, E(h) + O(1/N^2) \\ \langle 0 | s_n^z | 0 \rangle &= (2\pi)(1/h)[E(h) + (h^2 - 1)\, K(h)] + O(1/N^2) \end{aligned} \tag{A19}$$

where

$$K(x) = \int_0^{\pi/2} (1 - x^2 \sin^2 \alpha)^{-1/2} \, d\alpha \tag{A20}$$

is the complete elliptic integral of the first kind. Equations (A18) and (A19) are valid for periodic and for antiperiodic BCs as well. For the first excited state for $h > 1$ these equations are modified by a $2/N$ correction term:

$$\begin{aligned} \langle 1 | s_n^x s_{n+1}^x | 1 \rangle &= \langle 0 | s_n^x s_{n+1}^x | 0 \rangle + 2/N + O(1/N^2) \\ \langle 1 | s_n^z | 1 \rangle &= \langle 0 | s_n^z | 0 \rangle - 2/N + O(1/N^2) \end{aligned} \tag{A21}$$

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